Selected Problems in Dynamics: Stability of a Spinning Body

-and-

Derivation of the Lagrange Points

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Outline

- Stability of a Spinning Body

- Derivation of the Lagrange Points

TEAS Employees: Please log this time on the Training Tracker

Stability of a Spinning Body

- Euler's Second Law:

$$M = \frac{d}{dt}\left(I\omega\right)$$

Moments of Torque → M

Inertia Matrix → I

Angular Velocity → ω

(good for an inertial coordinate system)

Problem: Inertia matrix "*I*" of a rotating body changes in an inertial coordinate system

Euler's Second Law (2)

- In a body-fixed coordinate system:

$$M = \frac{d}{dt}\left(I\omega\right)_{body} + \omega \times \left(I\omega\right)$$

Moments of Torque (M)

Inertia Matrix (I)

Angular Velocity (ω)

- Inertia Matrix:
  - Describes distribution of mass on the body

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Euler's Second Law (3)

- Planes of Symmetry:
  - x-y plane: $I_{xz} = I_{yz} = 0$
  - x-z plane: $I_{xy} = I_{yz} = 0$
  - y-z plane: $I_{xz} = I_{xz} = 0$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

- For a freely-spinning body:

$$M = 0$$

$$\frac{d}{dt}\left(I\omega\right)_{body} + \omega \times \left(I\omega\right) = 0$$

Euler's Second Law (4)

- Rigid body: $$\frac{d}{dt}\left(I\omega\right)_{body} + \omega \times \left(I\omega\right) = 0$$
  - "*I*" matrix is constant in body coordinates

$$I\frac{d\omega}{dt} + \omega \times \left(I\omega\right) = 0$$

- Do the matrix multiplication:

$$I\omega = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{bmatrix}$$

Euler's Second Law (5)

- Do the cross product:

$$\omega\times(I\omega)=\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\times\begin{bmatrix}I_{xx}\omega_x\\ I_{yy}\omega_y\\ I_{zz}\omega_z\end{bmatrix}=\begin{bmatrix}\omega_y I_{zz}\omega_z-\omega_z I_{yy}\omega_y\\ \omega_z I_{xx}\omega_x-\omega_x I_{zz}\omega_z\\ \omega_x I_{yy}\omega_y-\omega_y I_{xx}\omega_x\end{bmatrix}$$

- Result:

$$\begin{bmatrix}I_{xx} & 0 & 0\\ 0 & I_{yy} & 0\\ 0 & 0 & I_{zz}\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}+\begin{bmatrix}\omega_y I_{zz}\omega_z-\omega_z I_{yy}\omega_y\\ \omega_z I_{xx}\omega_x-\omega_x I_{zz}\omega_z\\ \omega_x I_{yy}\omega_y-\omega_y I_{xx}\omega_x\end{bmatrix}=0$$

Euler's Second Law (6)

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} + \begin{bmatrix} \omega_y I_{zz} \omega_z - \omega_z I_{yy} \omega_y \\ \omega_z I_{xx} \omega_x - \omega_x I_{zz} \omega_z \\ \omega_x I_{yy} \omega_y - \omega_y I_{xx} \omega_x \end{bmatrix} = 0$$

- As scalar equations:

$$I_{xx} \frac{d\omega_x}{dt} + \left(I_{zz} - I_{yy}\right)\omega_y \omega_z = 0$$

$$I_{yy} \frac{d\omega_y}{dt} + \left(I_{xx} - I_{zz}\right)\omega_z \omega_x = 0$$

$$I_{zz} \frac{d\omega_z}{dt} + \left(I_{yy} - I_{xx}\right)\omega_x \omega_y = 0$$

Small Perturbations:

- Assume a primary rotation about one axis:
  - Remaining angular velocities are small

$$\omega_x = \Omega + \omega'_x$$

$$\omega_y = \omega'_y \qquad\qquad |\omega'_x|, |\omega'_y|, |\omega'_z| << \Omega$$

$$\omega_z = \omega'_z$$

Small Perturbations (2)

- Equations:

$$I_{xx}\frac{d\omega'_x}{dt}+\left(I_{zz}-I_{yy}\right)\omega'_y\,\omega'_z=0$$

$$I_{yy}\frac{d\omega'_y}{dt}+\left(I_{xx}-I_{zz}\right)\omega'_z\left(\Omega+\omega'_x\right)=0$$

$$I_{zz}\frac{d\omega'_z}{dt}+\left(I_{yy}-I_{xx}\right)\left(\Omega+\omega'_x\right)\omega'_y=0$$

Small Perturbations (3)

- Linearize:
  - Neglect products of small quantities

$$I_{xx} \frac{d\omega'_x}{dt} = 0$$

$$I_{yy} \frac{d\omega'_y}{dt} + \Omega\left(I_{xx} - I_{zz}\right)\omega'_z = 0$$

$$I_{zz} \frac{d\omega'_z}{dt} + \Omega\left(I_{yy} - I_{xx}\right)\omega'_y = 0$$

Small Perturbations (4)

- Combine the last two equations:

$$I_{yy}\frac{d\omega'_y}{dt}+\Omega\left(I_{xx}-I_{zz}\right)\omega'_z=0$$

$$I_{zz}\frac{d\omega'_z}{dt}+\Omega\left(I_{yy}-I_{xx}\right)\omega'_y=0$$

$$\frac{d\omega'_y}{dt}=-\Omega\frac{\left(I_{xx}-I_{zz}\right)}{I_{yy}}\omega'_z$$

switched

$$\frac{d\omega'_z}{dt}=\Omega\frac{\left(I_{xx}-I_{yy}\right)}{I_{zz}}\omega'_y$$

Small Perturbations (4)

- Combine the last two equations:

$$I_{yy}\frac{d\omega'_y}{dt}+\Omega\left(I_{xx}-I_{zz}\right)\omega'_z=0$$

$$I_{zz}\frac{d\omega'_z}{dt}+\Omega\left(I_{yy}-I_{xx}\right)\omega'_y=0$$

$$\frac{d^2\omega'_y}{dt^2}=-\Omega\frac{\left(I_{xx}-I_{zz}\right)}{I_{yy}}\frac{d\omega'_z}{dt}$$

$$\frac{d^2\omega'_z}{dt^2}=\Omega\frac{\left(I_{xx}-I_{yy}\right)}{I_{zz}}\frac{d\omega'_y}{dt}$$

Small Perturbations (4)

- Combine the last two equations:

$$I_{yy}\frac{d\omega'_y}{dt}+\Omega\left(I_{xx}-I_{zz}\right)\omega'_z=0$$

$$I_{zz}\frac{d\omega'_z}{dt}+\Omega\left(I_{yy}-I_{xx}\right)\omega'_y=0$$

$$\frac{d^2\omega'_y}{dt^2}+\Omega^2\frac{\left(I_{xx}-I_{zz}\right)\left(I_{xx}-I_{yy}\right)}{I_{yy}I_{zz}}\omega'_y=0$$

$$\frac{d^2\omega'_z}{dt^2}+\Omega^2\frac{\left(I_{xx}-I_{zz}\right)\left(I_{xx}-I_{yy}\right)}{I_{yy}I_{zz}}\omega'_z=0$$

Small Perturbations (5)

$$\frac{d^2\omega'_y}{dt^2}+\Omega^2\frac{(I_{xx}-I_{zz})(I_{xx}-I_{yy})}{I_{yy}I_{zz}}\omega'_y=0$$

$$\frac{d^2\omega'_z}{dt^2}+\Omega^2\frac{(I_{xx}-I_{zz})(I_{xx}-I_{yy})}{I_{yy}I_{zz}}\omega'_z=0$$

- Case 1: $I_{xx} > I_{yy}, I_{zz}$ or $I_{xx} < I_{yy}, I_{zz}$:

$$\Omega^2\frac{(I_{xx}-I_{zz})(I_{xx}-I_{yy})}{I_{yy}I_{zz}}>0$$

  - Sinusoidal solutions for ω'_y, ω'_z
  - Rotation is stable

Small Perturbations (6)

$$\frac{d^2\omega'_y}{dt^2} + \Omega^2 \frac{(I_{xx} - I_{zz})(I_{xx} - I_{yy})}{I_{yy}I_{zz}}\omega'_y = 0$$

$$\frac{d^2\omega'_z}{dt^2} + \Omega^2 \frac{(I_{xx} - I_{zz})(I_{xx} - I_{yy})}{I_{yy}I_{zz}}\omega'_z = 0$$

- Case 2: $I_{zz} > I_{xx} > I_{yy}$ or $I_{zz} < I_{xx} < I_{yy}$:

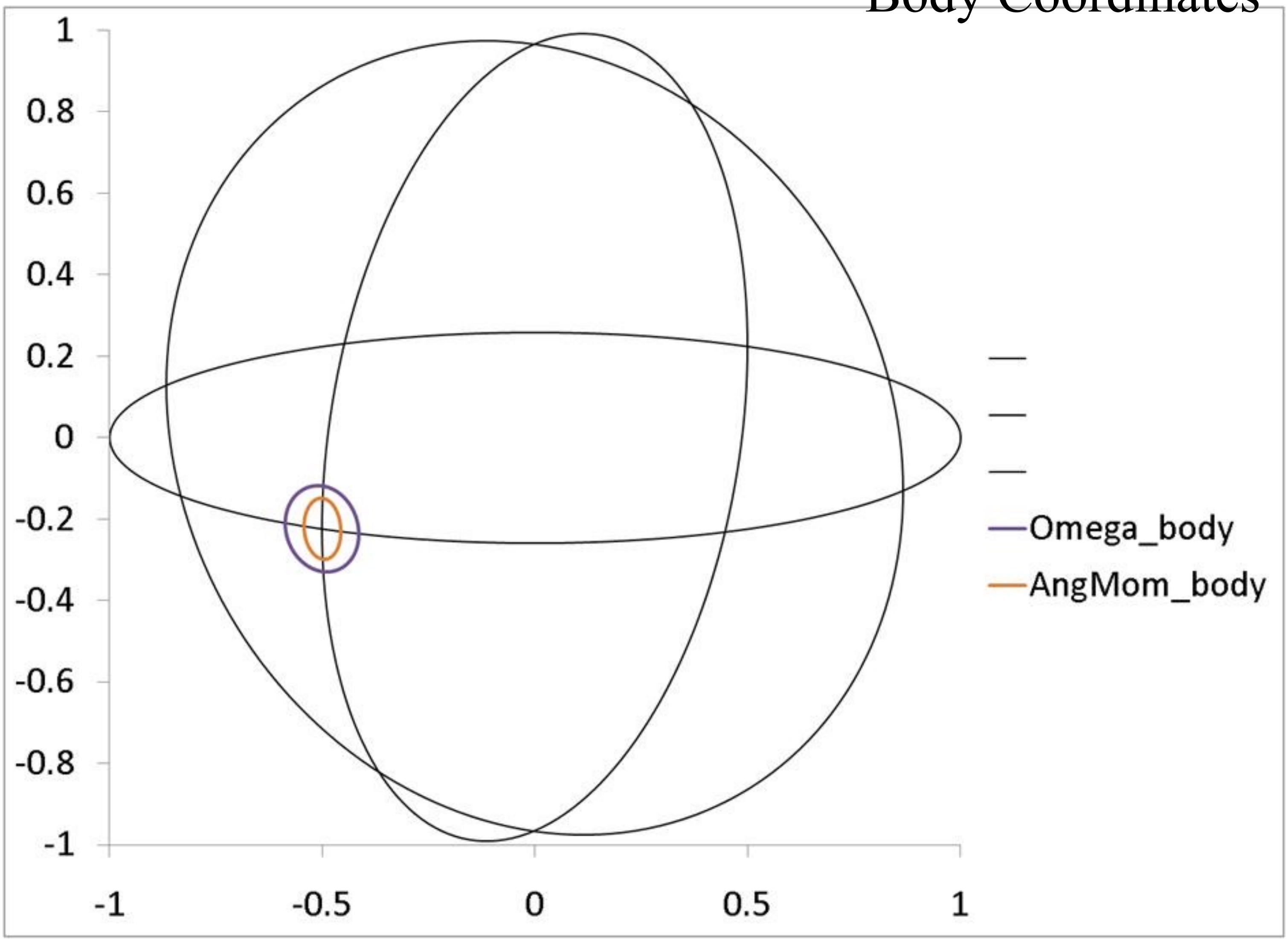
$$\Omega^2 \frac{(I_{xx} - I_{zz})(I_{xx} - I_{yy})}{I_{yy}I_{zz}} < 0$$

  - Hyperbolic solutions for ω'_y, ω'_z
  - Rotation is unstable

Numerical Solution

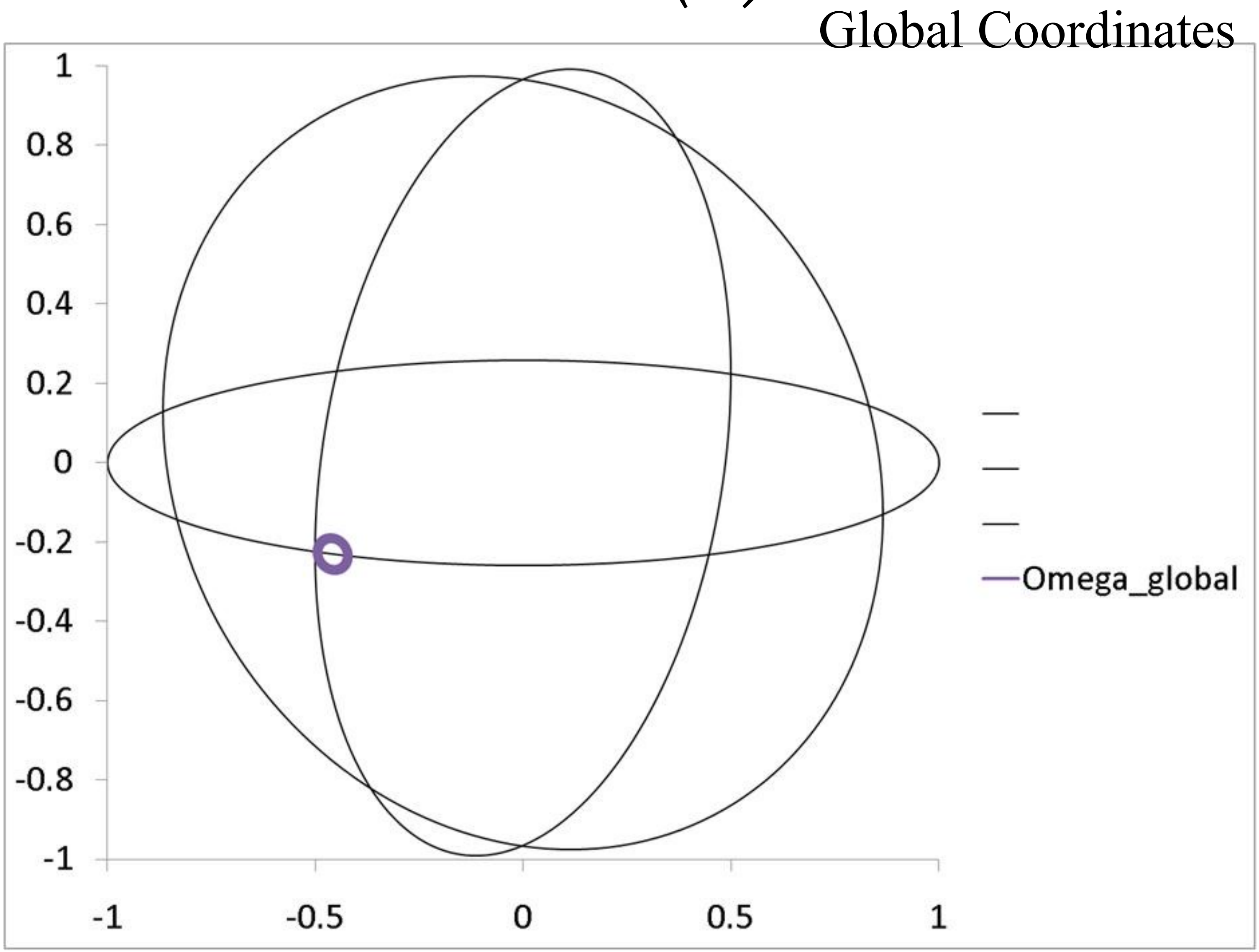
$I_{xx} = 1.0$
$I_{yy} = 0.5$
$I_{zz} = 0.7$

Body Coordinates



Numerical Solution (2)

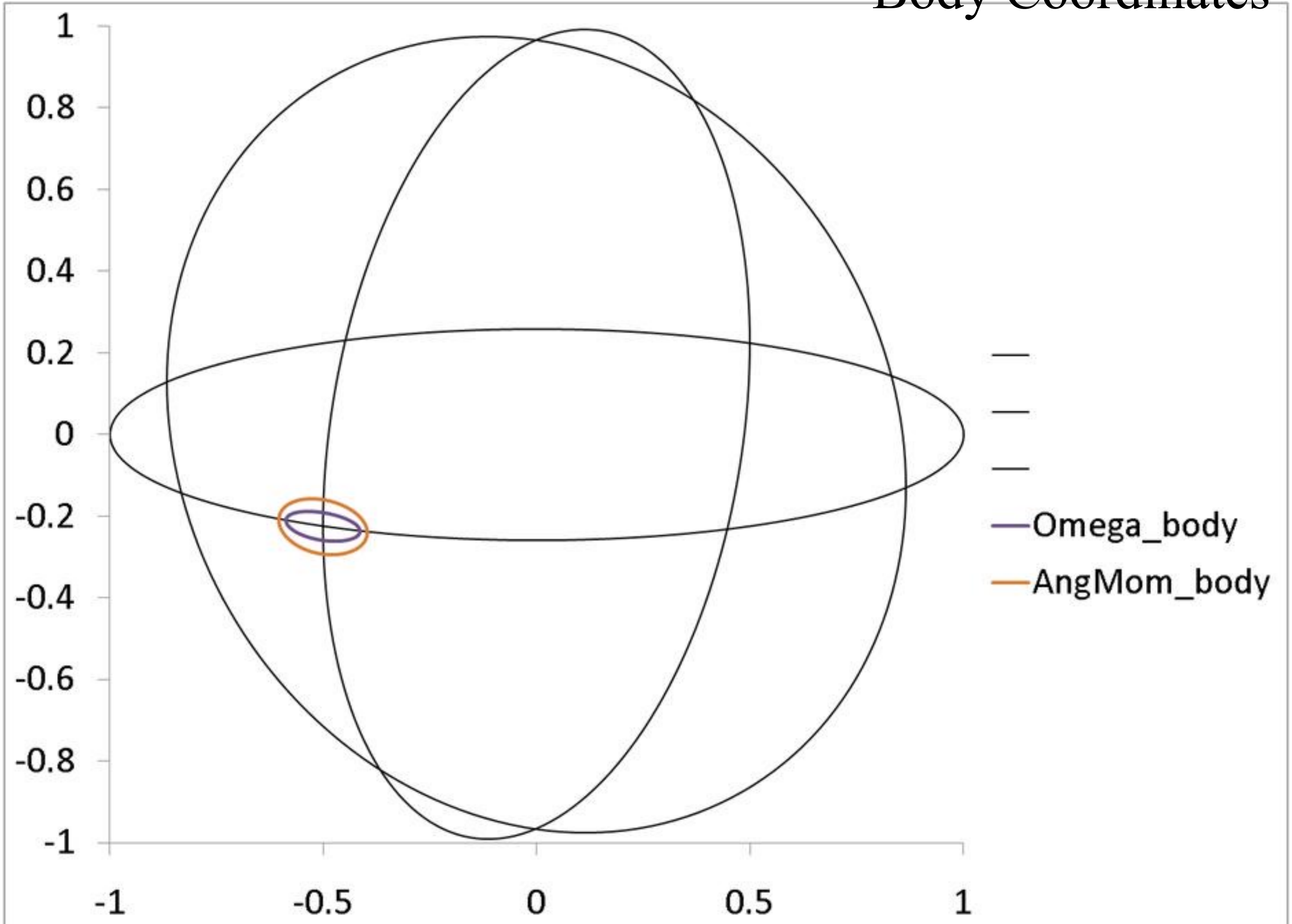
$I_{xx} = 1.0$
$I_{yy} = 0.5$
$I_{zz} = 0.7$



Numerical Solution (3)

$I_{xx} = 1.0$
$I_{yy} = 1.2$
$I_{zz} = 2.0$

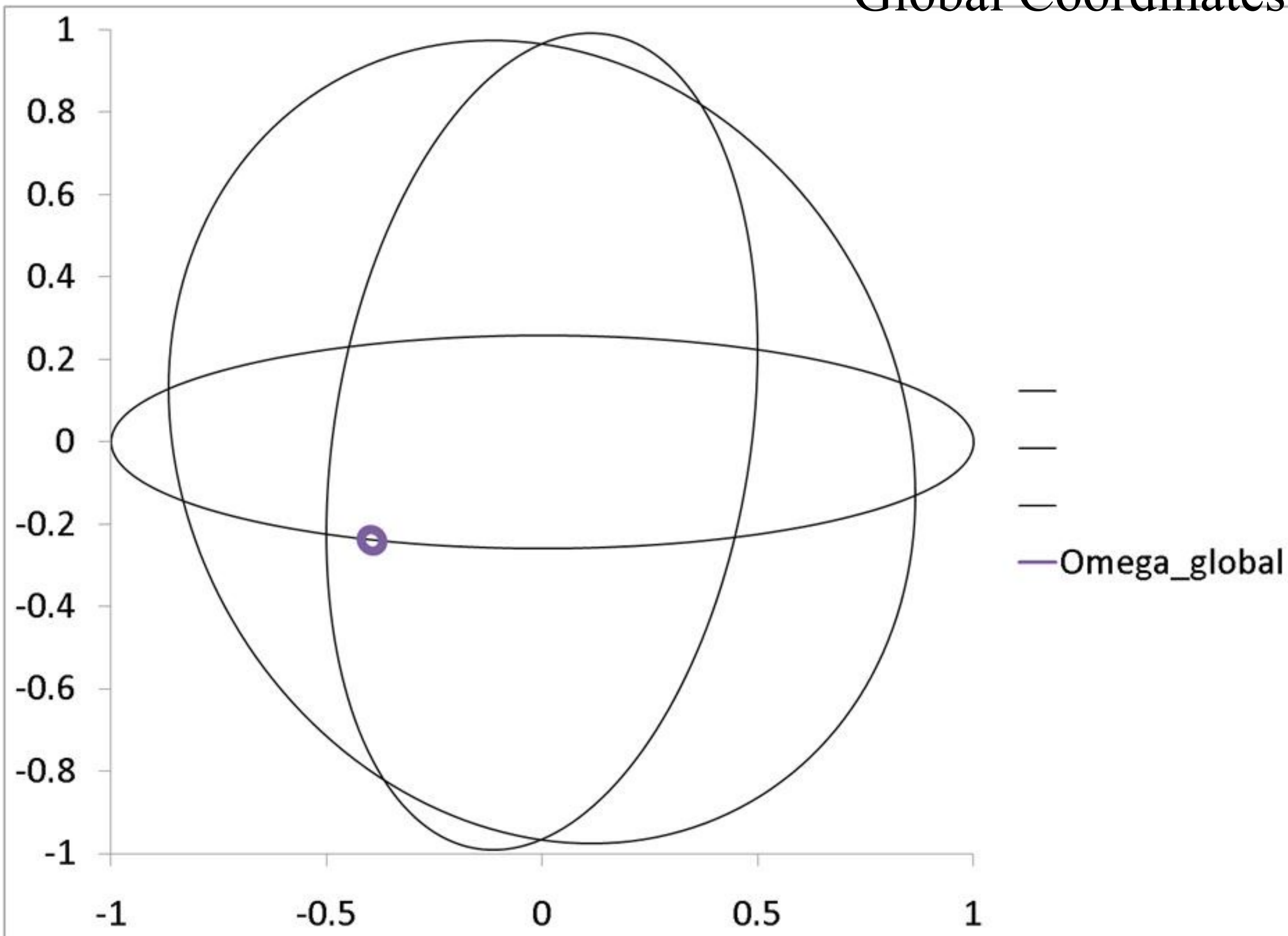
Body Coordinates



Numerical Solution (4)

Global Coordinates

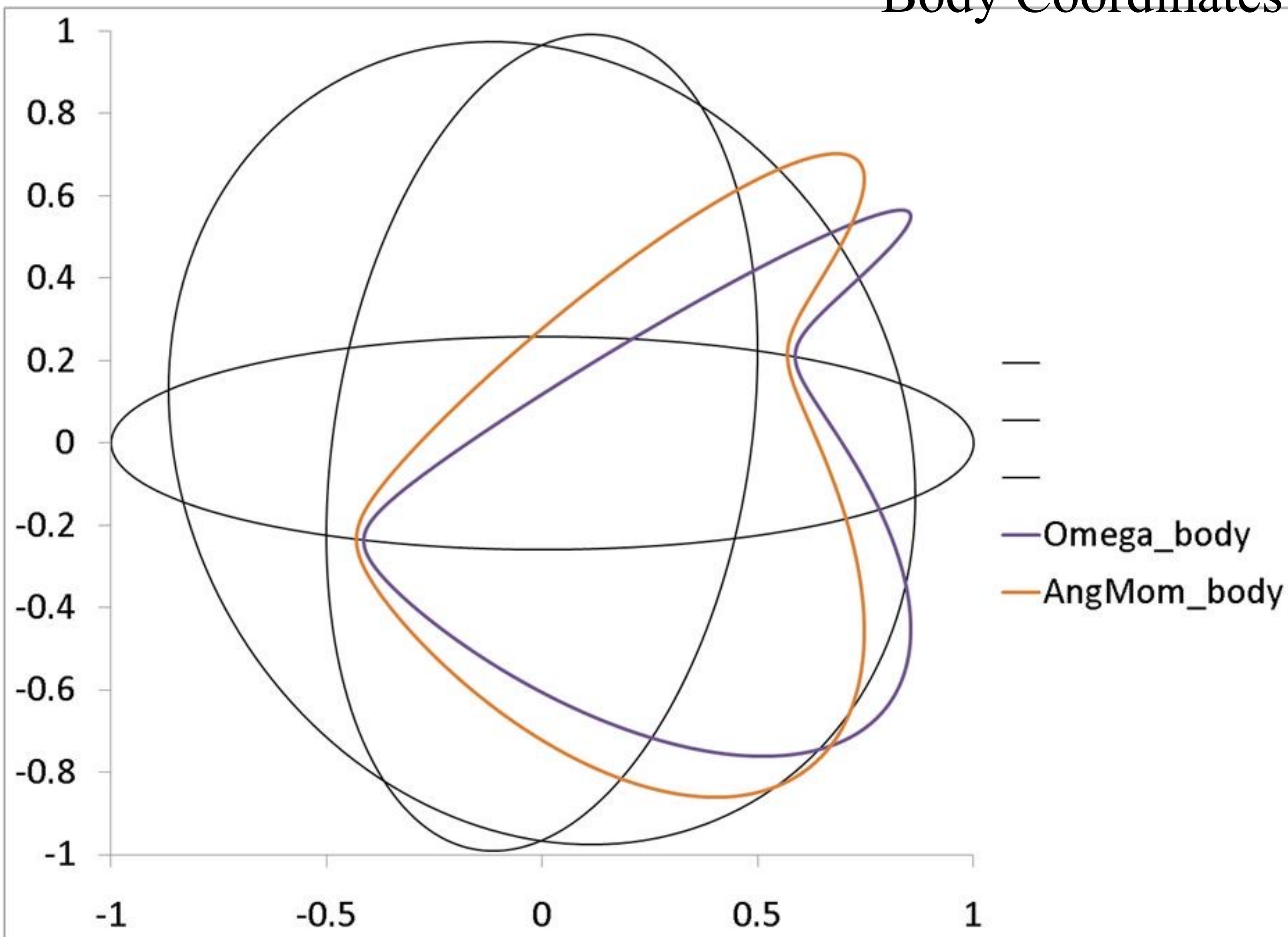
$I_{xx} = 1.0$
$I_{yy} = 1.2$
$I_{zz} = 2.0$



Numerical Solution (5)

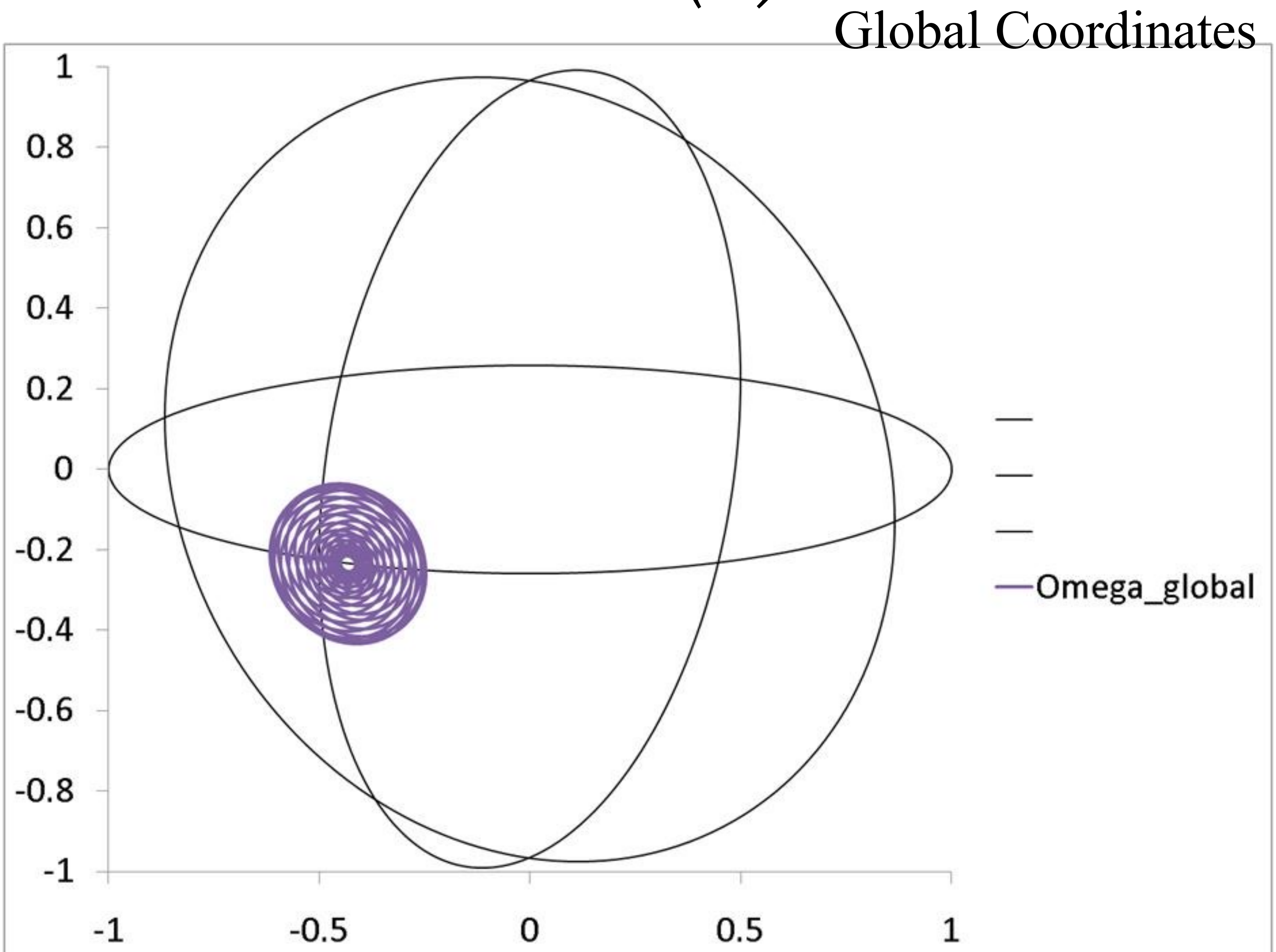
$I_{xx} = 1.0$
$I_{yy} = 0.8$
$I_{zz} = 1.2$

Body Coordinates



Numerical Solution (6)

$I_{xx} = 1.0$
$I_{yy} = 0.8$
$I_{zz} = 1.2$

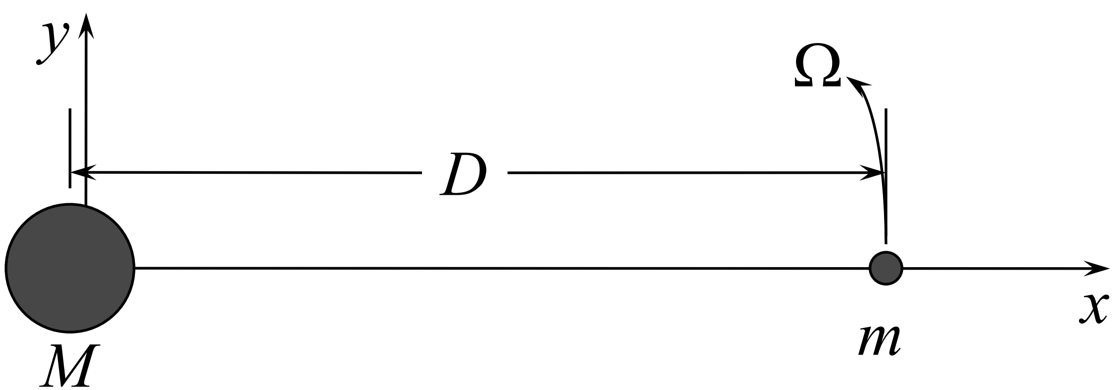


Spinning Body: Summary

- Rotation about axis with largest inertia: Stable
- Rotation about axis with smallest inertia: Stable
- Rotation about axis with in-between inertia: Unstable

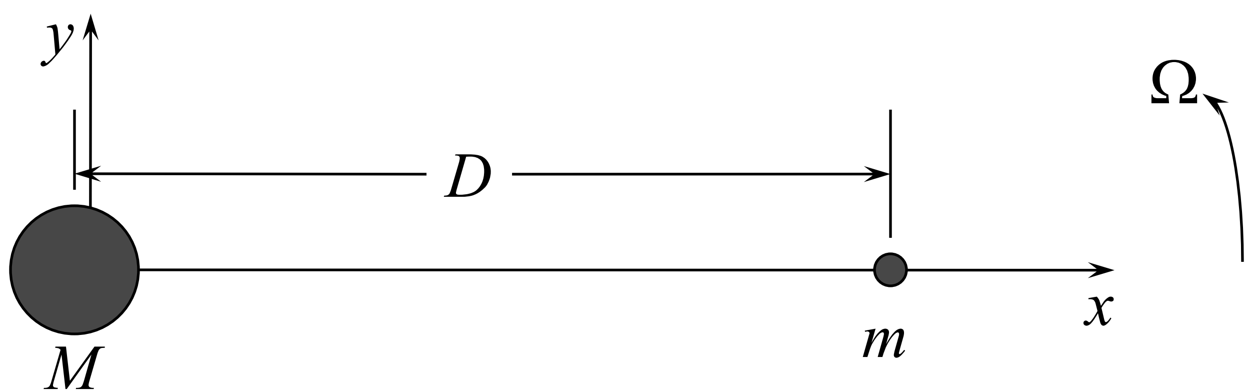
Derivation of the Lagrange Points

- Two-Body Problem:
  - Lighter body orbiting around a heavier body
    (Let's stick with a circular orbit)
  - Actually both bodies orbit around the barycenter



Two-Body Problem (2)

- Rotating reference frame:
  - Origin at barycenter of two-body system
  - Angular velocity matches orbital velocity of bodies
  - Bodies are stationary in the rotating reference frame



Two-Body Problem (3)

- Force Balance:

Body "M"

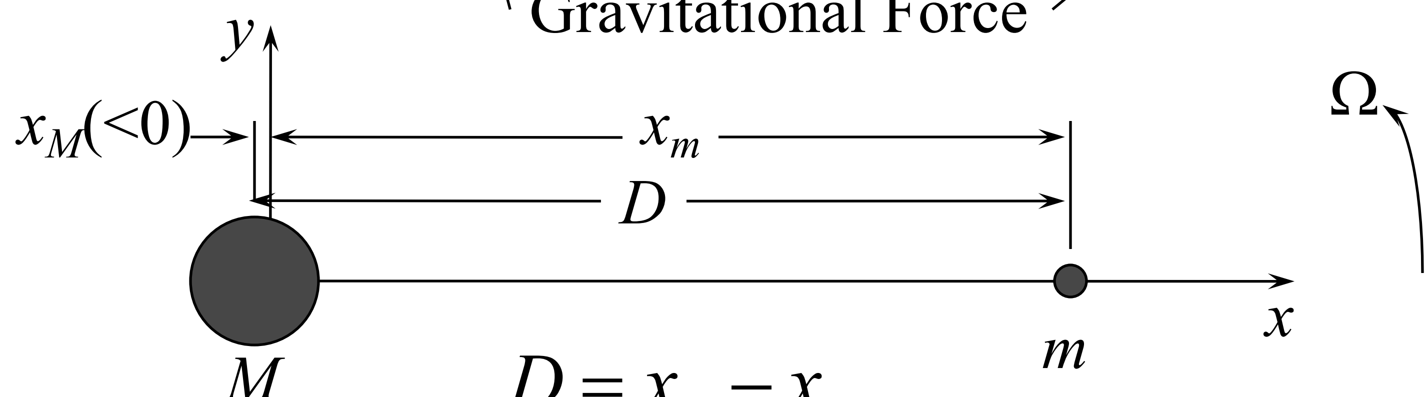
$$Mx_M\Omega^2 + \frac{GMm}{D^2} = 0$$

Body "m"

$$mx_m\Omega^2 - \frac{GMm}{D^2} = 0$$

Centrifugal Force

Gravitational Force



$$D = x_m - x_M$$

Two-Body Problem (4)

$$Mx_M\Omega^2 + \frac{GMm}{D^2} = 0$$

$$mx_m\Omega^2 - \frac{GMm}{D^2} = 0$$

$$D = x_m - x_M$$

- Parameters:
  - M – mass of large body
  - m – mass of small body
  - D – distance between bodies
- Unknowns:
  - x_M – x-coordinate of large body
  - x_m – x-coordinate of small body
  - Ω – angular velocity of coordinate system
- Three equations, three unknowns

In case anybody wants to see the math …

$$Mx_M\Omega^2 + \frac{GMm}{D^2} = 0$$

$$mx_m\Omega^2 - \frac{GMm}{D^2} = 0$$

$$D = x_m - x_M$$

$$x_M\Omega^2 + \frac{Gm}{D^2} = 0$$

$$x_m\Omega^2 - \frac{GM}{D^2} = 0$$

$$\Omega^2 = -\frac{Gm}{x_M D^2} = \frac{GM}{x_m D^2}$$

$$-\frac{m}{x_M} = \frac{M}{x_m}$$

$$-mx_m = Mx_M$$

$$D = x_m + \frac{m}{M}x_m = \frac{M+m}{M}x_m$$

$$x_m = \frac{M}{M+m}D$$

$$x_M = \frac{-m}{M+m}D$$

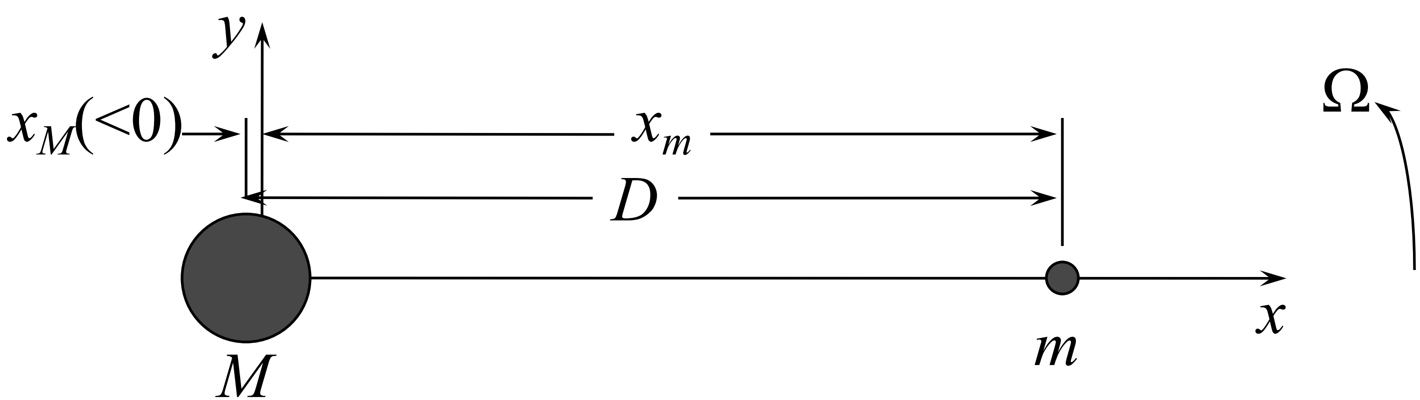
$$\Omega^2 = \frac{GM}{x_m D^2} = \frac{G(M+m)}{D^3}$$

Two-Body Problem (5)

- Solutions:

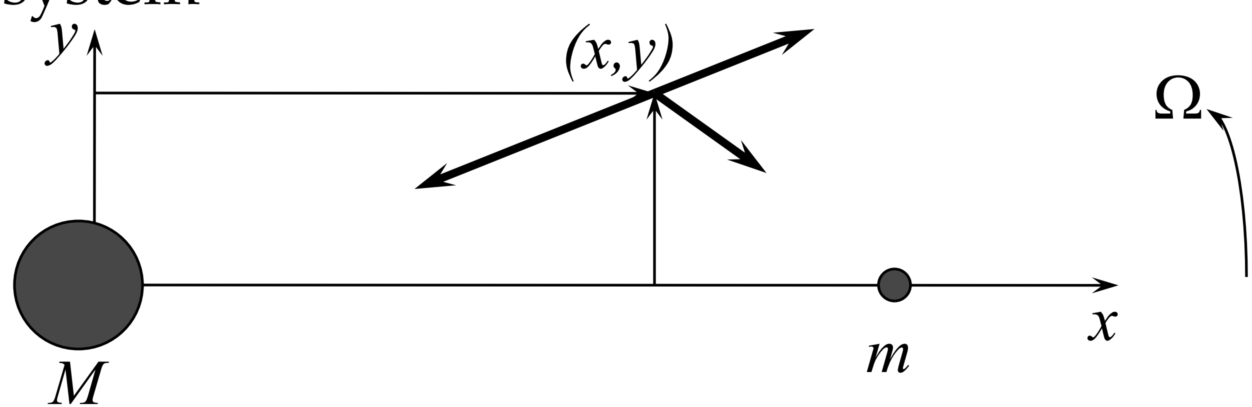
$$x_m = \frac{M}{M+m} D \qquad \Omega^2 = \frac{G(M+m)}{D^3}$$

$$x_M = -\frac{m}{M+m} D$$



Lagrange Points

- Points at which net gravitational and centrifugal forces are zero
  - Gravitational attraction to body "*M*"
  - Gravitational attraction to body "*m*"
  - Centrifugal force caused by rotating coordinate system



Lagrange Points (2)

- Sum of Forces: particle at (x,y) of mass "μ"
  - x-direction:

$$\frac{-GM\mu\left(x-x_M\right)}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{Gm\mu\left(x-x_m\right)}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+\mu x\Omega^2=0$$

  - y-direction:

$$\frac{-GM\mu y}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{Gm\mu y}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+\mu y\Omega^2=0$$

Lagrange Points (3)

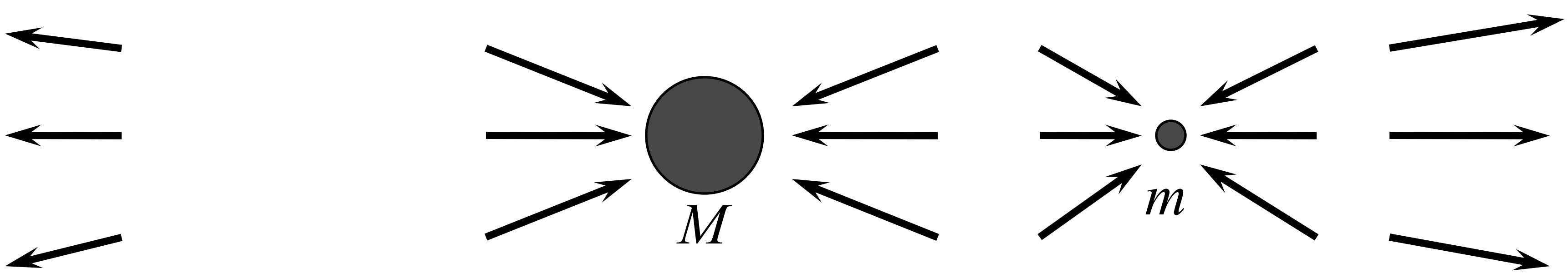
- Simplifications:
  - Divide by mass "μ" of particle
  - Substitute for "Ω^2" and divide by "G"

$$\frac{-M\left(x-x_M\right)}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{m\left(x-x_m\right)}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+x\frac{\left(M+m\right)}{D^3}=0$$

$$\frac{-My}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{my}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+y\frac{\left(M+m\right)}{D^3}=0$$

Lagrange Points (4)

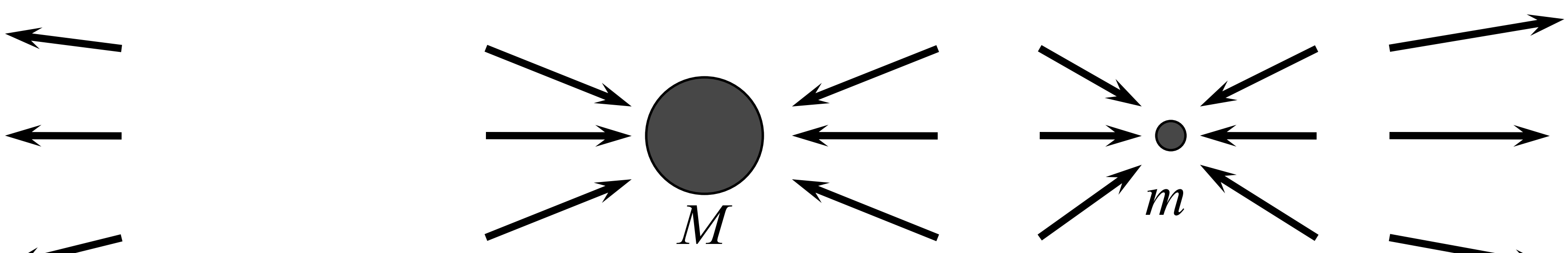
- Qualitative considerations:
  - Far enough from the system, centrifugal force will overwhelm everything else ➔Net force will be away from the origin
  - Close enough to "*M*", gravitational attraction to that body will overwhelm everything else
  - Close enough to "*m*", gravitational attraction to that body will overwhelm everything else



Lagrange Points (5)

❖ Qualitative considerations (2)

- Close to "*m*" but beyond it, the attraction to "*m*" will balance the centrifugal force ➔ search for a Lagrange point there
- On the side of "*M*" opposite "*m*", the attraction to "*M*" will balance the centrifugal force ➔ search for a Lagrange point there
- Between "*M*" and "*m*" their gravitational attractions will balance ➔ search for a Lagrange point there



Lagrange Points (6)

- Some Solutions:
  - Note that "*y*" equation is uniquely satisfied if $y = 0$ (but there may be nonzero solutions also)

$$\frac{-My}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{my}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+y\frac{\left(M+m\right)}{D^3}=0$$

Lagrange Points (7)

- Some Solutions:
  - "x" equation when "y" is zero:

$$\frac{-M\left(x-x_M\right)}{\left[\left(x-x_M\right)^2\right]^{3/2}}-\frac{m\left(x-x_m\right)}{\left[\left(x-x_m\right)^2\right]^{3/2}}+x\frac{\left(M+m\right)}{D^3}=0$$

  - Note: $\left[\left(x-x_M\right)^2\right]^{3/2}=\left|x-x_M\right|^3 \overset{?}{\neq} \left(x-x_M\right)^3$

$$-MD^3\left(x-x_m\right)\left|x-x_m\right|-mD^3\left(x-x_M\right)\left|x-x_M\right|$$
$$+x\left(M+m\right)\left(x-x_m\right)\left|x-x_m\right|\left(x-x_M\right)\left|x-x_M\right|=0$$

Lagrange Points (8)

$$\frac{-M\left(x-x_M\right)}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{m\left(x-x_m\right)}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+x\frac{\left(M+m\right)}{D^3}=0$$

$$\frac{-My}{\left[\left(x-x_M\right)^2+y^2\right]^{3/2}}-\frac{my}{\left[\left(x-x_m\right)^2+y^2\right]^{3/2}}+y\frac{\left(M+m\right)}{D^3}=0$$

- Oversimplification: Let "m" = 0
  - $x_M = 0$

$$\frac{-Mx}{\left[x^2+y^2\right]^{3/2}}+x\frac{M}{D^3}=0 \qquad \frac{-My}{\left[x^2+y^2\right]^{3/2}}+y\frac{M}{D^3}=0$$

- Solutions: $\left[x^2+y^2\right]^{3/2}=D^3$
  - Anywhere on the orbit of "m"

Lagrange Points (9)

$$-MD^3\left(x-x_m\right)\left|x-x_m\right|-mD^3\left(x-x_M\right)\left|x-x_M\right|$$
$$+x\left(M+m\right)\left(x-x_m\right)\left|x-x_m\right|\left(x-x_M\right)\left|x-x_M\right|=0$$

- Perturbation: $m=\varepsilon M \quad 0<\varepsilon<<1$

  $x_M=-\varepsilon D \quad x_m=\left(1-\varepsilon\right)D$

- For the y = 0 case:

$$-D^3\left(x-\left(1-\varepsilon\right)D\right)\left|x-\left(1-\varepsilon\right)D\right|$$
$$-\varepsilon D^3\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|$$
$$+x\left(1+\varepsilon\right)\left(x-\left(1-\varepsilon\right)D\right)\left|x-\left(1-\varepsilon\right)D\right|\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|=0$$

Lagrange Points (10)

$$-MD^3\left(x-x_m\right)\left|x-x_m\right|-mD^3\left(x-x_M\right)\left|x-x_M\right|$$
$$+x\left(M+m\right)\left(x-x_m\right)\left|x-x_m\right|\left(x-x_M\right)\left|x-x_M\right|=0$$

$$-D^3\left(x-\left(1-\varepsilon\right)D\right)\left|x-\left(1-\varepsilon\right)D\right|-\varepsilon D^3\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|$$
$$+x\left(1+\varepsilon\right)\left(x-\left(1-\varepsilon\right)D\right)\left|x-\left(1-\varepsilon\right)D\right|\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|=0$$

- For the *y* = 0 case (cont'd):
  - Lagrange Points L1,L2: near $x = D$:

$$x=D\left(1+\delta\right)$$

$$-D^5\left(\delta+\varepsilon\right)\left|\delta+\varepsilon\right|-\varepsilon D^5\left(1+\delta+\varepsilon\right)^2$$
$$+D^5\left(1+\delta\right)\left(1+\varepsilon\right)\left(\delta+\varepsilon\right)\left|\delta+\varepsilon\right|\left(1+\delta+\varepsilon\right)^2=0$$

L1, L2: y = 0, x = D(1 + δ)

$$-D^5(\delta+\varepsilon)|\delta+\varepsilon|-\varepsilon D^5(1+\delta+\varepsilon)^2$$

$$+D^5(1+\delta)(1+\varepsilon)(\delta+\varepsilon)|\delta+\varepsilon|(1+\delta+\varepsilon)^2=0$$

- Collecting terms:

$$(\delta+\varepsilon)|\delta+\varepsilon|\left[-1+(1+\delta)(1+\varepsilon)(1+\delta+\varepsilon)^2\right]$$

$$-\varepsilon(1+\delta+\varepsilon)^2=0$$

- Multiplying things out:

$$(\delta+\varepsilon)|\delta+\varepsilon|\begin{bmatrix}3\delta+3\delta^2+\delta^3+3\varepsilon+7\delta\varepsilon+5\delta^2\varepsilon+\delta^3\varepsilon\\+3\varepsilon^2+5\delta\varepsilon^2+2\delta^2\varepsilon^2+\varepsilon^3+\delta\varepsilon^3\end{bmatrix}$$

$$=\varepsilon\left(1+2\delta+2\varepsilon+\delta^2+2\delta\varepsilon+\varepsilon^2\right)$$

L1, L2 (2): y = 0, x = D(1 + δ)

- Balancing terms:

Order of magnitude δ^3

$$(\delta+\varepsilon)|\delta+\varepsilon|\left[\begin{array}{l}3\delta+3\delta^2+\delta^3+3\varepsilon+7\delta\varepsilon+5\delta^2\varepsilon+\delta^3\varepsilon\\+3\varepsilon^2+5\delta\varepsilon^2+2\delta^2\varepsilon^2+\varepsilon^3+\delta\varepsilon^3\end{array}\right]$$

$$=\varepsilon\left(1+2\delta+2\varepsilon+\delta^2+2\delta\varepsilon+\varepsilon^2\right)$$

Order of magnitude ε

- Linearizing:

$$3\delta^3 \cong \pm\varepsilon \qquad \delta \cong \pm\sqrt[3]{\frac{1}{3}\varepsilon}$$

Lagrange Points (11)

$$-D^3\left(x-(1-\varepsilon)D\right)\left|x-(1-\varepsilon)D\right|-\varepsilon D^3\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|$$

$$+x(1+\varepsilon)\left(x-(1-\varepsilon)D\right)\left|x-(1-\varepsilon)D\right|\left(x+\varepsilon D\right)\left|x+\varepsilon D\right|=0$$

- Lagrange Point L3: $y = 0$, near $x = -D$

$$x=-D(1+\delta)$$

$$(2+\delta-\varepsilon)\left|2+\delta-\varepsilon\right|+\varepsilon(1+\delta-\varepsilon)\left|1+\delta-\varepsilon\right|$$

$$-(1+\delta)(1+\varepsilon)(2+\delta-\varepsilon)\left|2+\delta-\varepsilon\right|(1+\delta-\varepsilon)\left|1+\delta-\varepsilon\right|=0$$

- Since $|\varepsilon|, |\delta| << 1 < 2$:

$$(2+\delta-\varepsilon)^2+\varepsilon(1+\delta-\varepsilon)^2$$

$$-(1+\delta)(1+\varepsilon)(2+\delta-\varepsilon)^2(1+\delta-\varepsilon)^2=0$$

L3: $y = 0,\ x = -D(1+\delta)$

$$(2+\delta-\varepsilon)^2+\varepsilon(1+\delta-\varepsilon)^2 - (1+\delta)(1+\varepsilon)(2+\delta-\varepsilon)^2(1+\delta-\varepsilon)^2 = 0$$

- Multiply out and linearize:

$$(4+4\delta-4\varepsilon)+\varepsilon - (1+\delta+\varepsilon)(4+4\delta-4\varepsilon)(1+2\delta-2\varepsilon) \cong 0$$

$$(4+4\delta-4\varepsilon)+\varepsilon - (4+4\delta+4\varepsilon+4\delta-4\varepsilon+8\delta-8\varepsilon) \cong 0$$

$$5\varepsilon-12\delta \cong 0 \qquad \delta \cong \frac{5}{12}\varepsilon$$

Lagrange Points (12)

- Summary for a Small Perturbation and $y = 0$: $m = \varepsilon M \quad 0 < \varepsilon << 1$

  $x_M = -\varepsilon D \quad x_m = (1-\varepsilon)D$

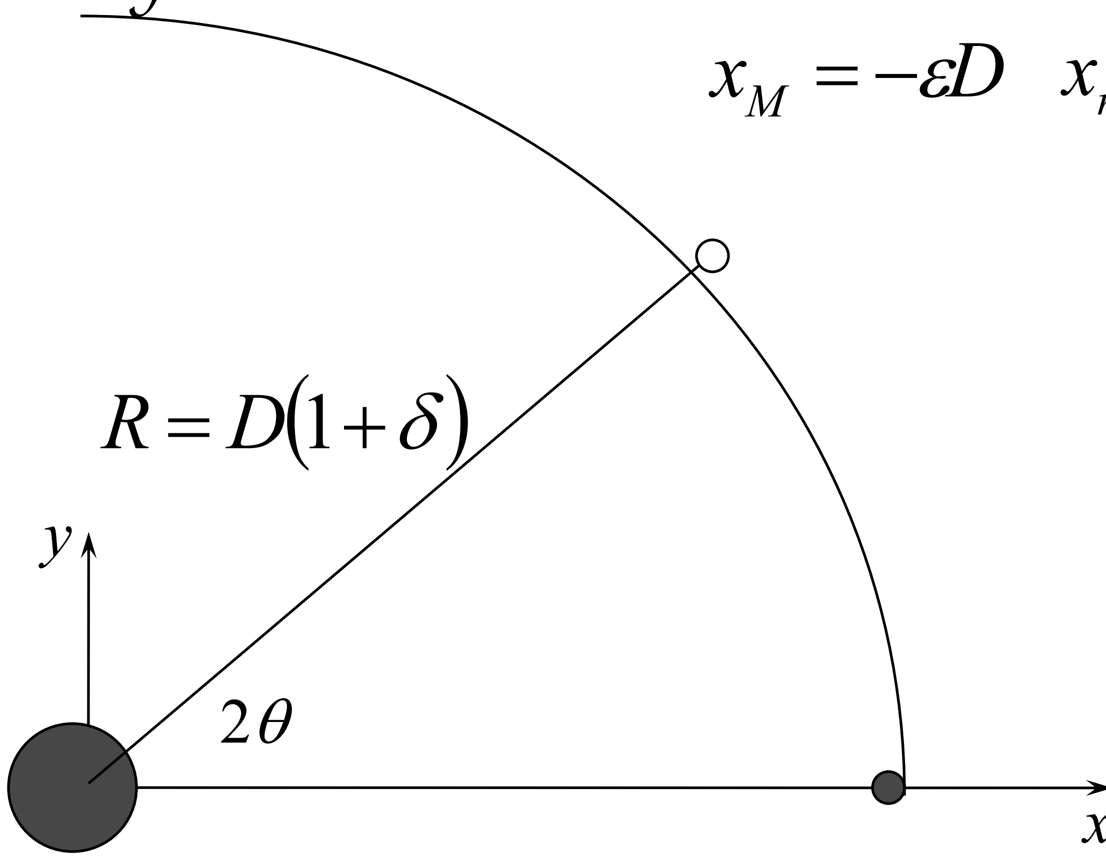
- Lagrange Points:
  - L1: $\rightarrow x \cong D\left(1 - \sqrt[3]{\frac{1}{3}\varepsilon}\right)$
  - L2: $\longrightarrow x \cong D\left(1 + \sqrt[3]{\frac{1}{3}\varepsilon}\right)$
  - L3: $\rightarrow x \cong -D\left(1 + \frac{5}{12}\varepsilon\right)$

Lagrange Points (13)

- What if y is not zero?

$$m = \varepsilon M \quad 0 < \varepsilon << 1$$

$$x_M = -\varepsilon D \quad x_m = (1-\varepsilon)D$$



Lagrange Points (14)

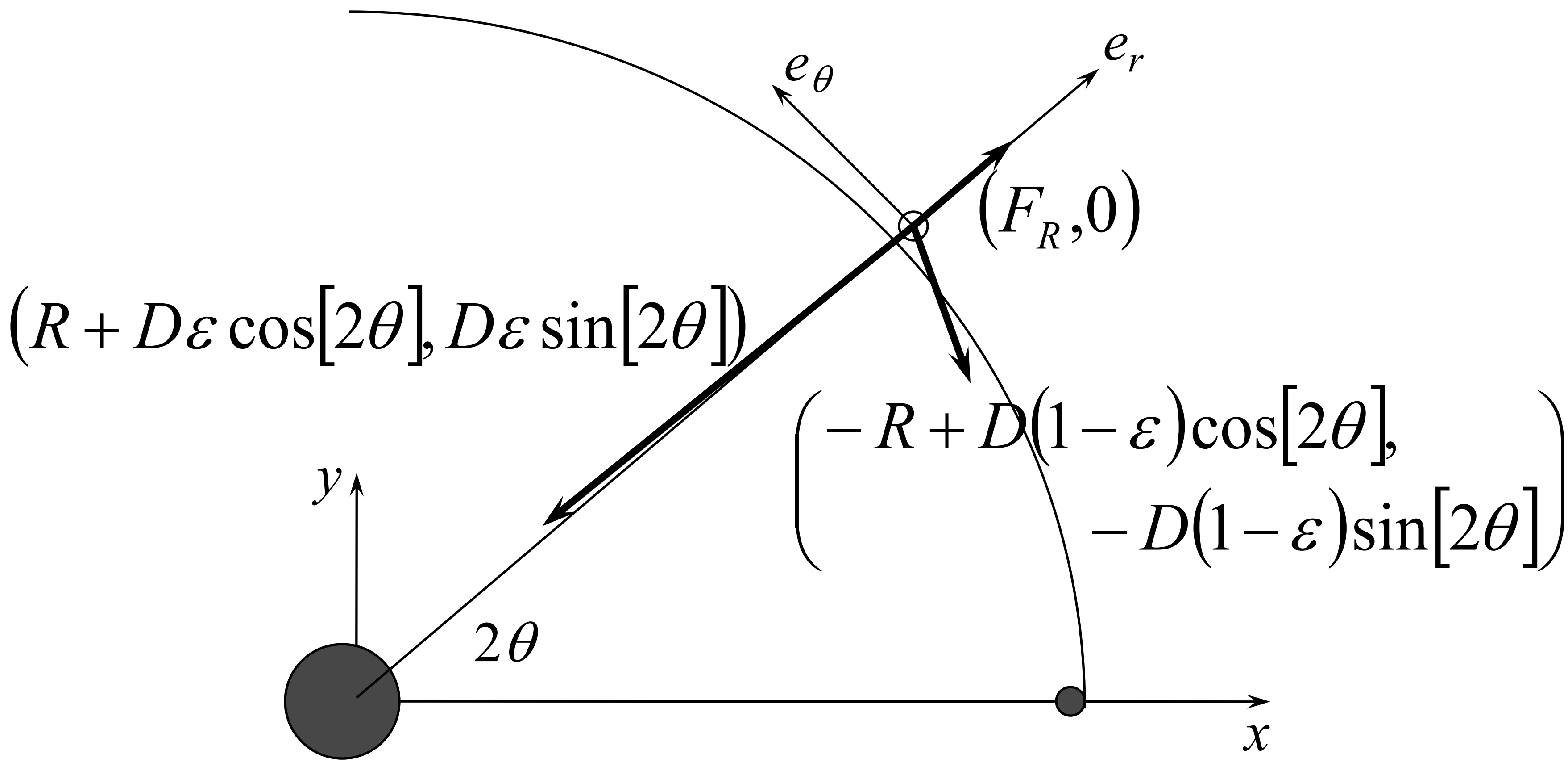
$$m = \varepsilon M \qquad 0 < \varepsilon << 1$$

$$x_M = -\varepsilon D$$

$$x_m = (1-\varepsilon)D$$

$$R = D(1+\delta)$$

- Polar Coordinates:



Lagrange Points (15)

- Consider the forces in the *R*-direction:
  - From *M*:

$$F_M = \frac{-GM\mu\left(D(1+\delta)+D\varepsilon\cos(2\theta)\right)}{\left\{\left[D(1+\delta)+D\varepsilon\cos(2\theta)\right]^2+\left[D\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - From *m*:

$$F_m = \frac{Gm\mu\left(-D(1+\delta)+D(1-\varepsilon)\cos(2\theta)\right)}{\left\{\left[-D(1+\delta)+D(1-\varepsilon)\cos(2\theta)\right]^2+\left[D(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - Centrifugal force:

$$F_C = \mu\Omega^2 D(1+\delta)$$

R-direction Forces

- Simplify:
  - From *M*:

$$F_M = \frac{-GM\mu\left(1+\delta+\varepsilon\cos(2\theta)\right)}{D^2\left\{\left[1+\delta+\varepsilon\cos(2\theta)\right]^2+\left[\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - From *m*:

$$F_m = \frac{-G\varepsilon M\mu\left(1+\delta-(1-\varepsilon)\cos(2\theta)\right)}{D^2\left\{\left[1+\delta-(1-\varepsilon)\cos(2\theta)\right]^2+\left[(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - Centrifugal force:

$$F_C = \frac{GM(1+\varepsilon)\mu}{D^2}(1+\delta)$$

R-direction Forces (2)

- Linearize in δ and ε:
  - From M:

$$F_M \cong \frac{-GM\mu\left(1+\delta+\varepsilon\cos(2\theta)\right)}{D^2\left[1+3\delta+3\varepsilon\cos(2\theta)\right]} \cong \frac{-GM\mu}{D^2}\left(1-2\delta-2\varepsilon\cos(2\theta)\right)$$

  - From m:

$$F_m \cong \frac{-G\varepsilon M\mu\left(1+\delta-(1-\varepsilon)\cos(2\theta)\right)}{D^2\left\{1+2\delta-2(1+\delta-\varepsilon)\cos(2\theta)+1-2\varepsilon\right\}^{3/2}} \cong \frac{-GM\mu}{2\sqrt{2}D^2}\varepsilon$$

  - Centrifugal force:

$$F_C \cong \frac{GM\mu}{D^2}\left(1+\delta+\varepsilon\right)$$

R-direction Forces (3)

- Balance the Forces:

$$-\frac{GM\mu}{D^2}\left(1-2\delta-2\varepsilon\cos(2\theta)\right)-\frac{GM\mu}{2\sqrt{2}D^2}\varepsilon+\frac{GM\mu}{D^2}\left(1+\delta+\varepsilon\right)\cong 0$$

$$-\left(1-2\delta-2\varepsilon\cos(2\theta)\right)-\frac{\varepsilon}{2\sqrt{2}}+\left(1+\delta+\varepsilon\right)\cong 0$$

$$1+\delta+\varepsilon\cong 1-2\delta-2\varepsilon\cos(2\theta)+\frac{\varepsilon}{2\sqrt{2}}$$

$$3\delta\cong-\varepsilon-2\varepsilon\cos(2\theta)+\frac{\varepsilon}{2\sqrt{2}}$$

$$\delta\cong\left(\frac{1}{2\sqrt{2}}-1-2\cos(2\theta)\right)\frac{\varepsilon}{3}$$

Lagrange Points (16)

- Consider the forces in the θ-direction:
  - From M:

$$F_M = \frac{GM\mu D\varepsilon \sin(2\theta)}{\left\{\left[D(1+\delta)+D\varepsilon\cos(2\theta)\right]^2+\left[D\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - From m:

$$F_m = \frac{-Gm\mu D(1-\varepsilon)\sin(2\theta)}{\left\{\left[-D(1+\delta)+D(1-\varepsilon)\cos(2\theta)\right]^2+\left[D(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

  - Centrifugal force is completely radial

$$F_C = 0$$

θ-Direction Forces

$$F_M = \frac{GM\mu D\varepsilon \sin(2\theta)}{\left\{\left[D(1+\delta) + D\varepsilon\cos(2\theta)\right]^2 + \left[D\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

$$F_m = \frac{-Gm\mu D(1-\varepsilon)\sin(2\theta)}{\left\{\left[-D(1+\delta) + D(1-\varepsilon)\cos(2\theta)\right]^2 + \left[D(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

- Simplify:

$$F_M = \frac{GM\mu\varepsilon \sin(2\theta)}{D^2\left\{\left[(1+\delta) + \varepsilon\cos(2\theta)\right]^2 + \left[\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

$$F_m = \frac{-GM\mu\varepsilon(1-\varepsilon)\sin(2\theta)}{D^2\left\{\left[(1+\delta) - (1-\varepsilon)\cos(2\theta)\right]^2 + \left[(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

θ-Direction Forces (2)

$$F_M = \frac{GM\mu\varepsilon\sin(2\theta)}{D^2\left\{\left[(1+\delta)+\varepsilon\cos(2\theta)\right]^2+\left[\varepsilon\sin(2\theta)\right]^2\right\}^{3/2}}$$

$$F_m = \frac{-GM\mu\varepsilon(1-\varepsilon)\sin(2\theta)}{D^2\left\{\left[(1+\delta)-(1-\varepsilon)\cos(2\theta)\right]^2+\left[(1-\varepsilon)\sin(2\theta)\right]^2\right\}^{3/2}}$$

- Linearize in δ and ε:

$$F_M \cong \frac{GM\mu\varepsilon\sin(2\theta)}{D^2\left\{1+3\delta+3\varepsilon\cos(2\theta)\right\}} \cong \frac{GM\mu\sin(2\theta)}{D^2}\varepsilon$$

$$F_m \cong \frac{-GM\mu\varepsilon(1-\varepsilon)\sin(2\theta)}{D^2\left\{1+2\delta-2(1+\delta-\varepsilon)\cos(2\theta)+1-2\varepsilon\right\}^{3/2}}$$

$$\cong \frac{-GM\mu\sin(2\theta)}{D^2\left\{2-2\cos(2\theta)\right\}^{3/2}}\varepsilon$$

θ-Direction Forces (3)

$$F_M \cong \frac{GM\mu\sin(2\theta)}{D^2}\varepsilon$$

$$F_m \cong \frac{-GM\mu\sin(2\theta)}{D^2\{2-2\cos(2\theta)\}^{3/2}}\varepsilon$$

- Simplifying further:

$$F_M \cong \frac{GM\mu\sin(2\theta)}{D^2}\varepsilon$$

$$F_m \cong \frac{-GM\mu\sin(2\theta)}{D^2 2\sqrt{2}\{1-[\cos^2(\theta)-\sin^2(\theta)]\}^{3/2}}\varepsilon$$

$$\cong \frac{-GM\mu\sin(2\theta)}{D^2 2\sqrt{2}\{2\sin^2(\theta)\}^{3/2}}\varepsilon \cong \frac{-GM\mu\sin(2\theta)}{8D^2|\sin^3(\theta)|}\varepsilon$$

θ-Direction Forces (4)

$$F_M \cong \frac{GM\mu\sin(2\theta)}{D^2}\varepsilon$$

$$F_m \cong \frac{-GM\mu\sin(2\theta)}{8D^2\left|\sin^3(\theta)\right|}\varepsilon$$

- Balance the two forces:

$$\frac{GM\mu\sin(2\theta)}{D^2}\varepsilon + \frac{-GM\mu\sin(2\theta)}{8D^2\left|\sin^3(\theta)\right|}\varepsilon \cong 0$$

$$\frac{GM\mu\sin(2\theta)}{D^2}\varepsilon \cong \frac{GM\mu\sin(2\theta)}{8D^2\left|\sin^3(\theta)\right|}\varepsilon$$

$$8\left|\sin^3(\theta)\right| \cong 1$$

$$\theta \cong \pm\frac{\pi}{6} \cong \pm 30^\circ \qquad 2\theta \cong \pm\frac{\pi}{3} \cong \pm 60^\circ$$

$$\sin(\theta) \cong \pm\frac{1}{2}$$

Lagrange Points (17)

- Solving for our radius: $R = D(1+\delta)$

$$\delta \cong \left(\frac{1}{2\sqrt{2}} - 1 - 2\cos(2\theta)\right)\frac{\varepsilon}{3} \qquad 2\theta \cong \pm\frac{\pi}{3} \cong \pm 60^\circ$$

$$R \cong D\left(1 + \left(\frac{1}{2\sqrt{2}} - 1 - 2\cos(2\theta)\right)\frac{\varepsilon}{3}\right)$$

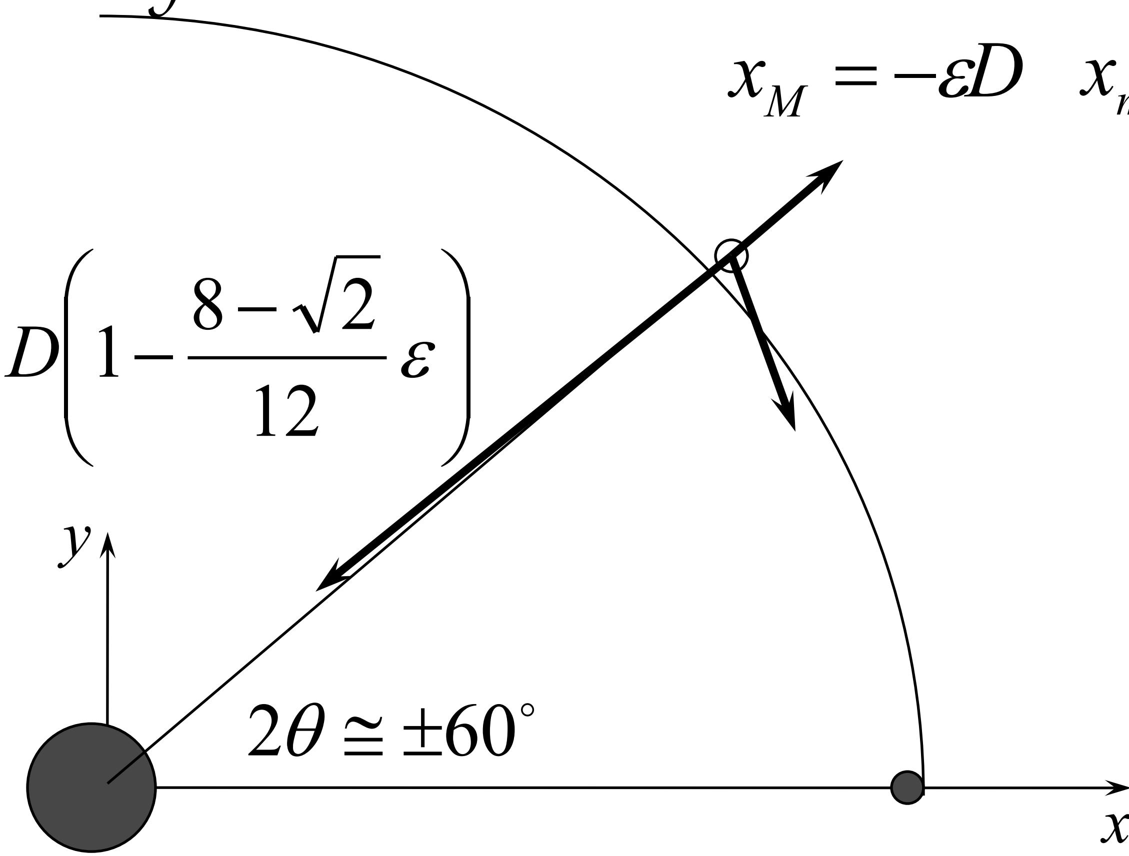
$$\cong D\left(1 + \left(\frac{1}{2\sqrt{2}} - 1 - 2 \bullet \frac{1}{2}\right)\frac{\varepsilon}{3}\right) \cong D\left(1 - \frac{8-\sqrt{2}}{12}\varepsilon\right)$$

Lagrange Points (13 redux)

- What if y is not zero?

$$m = \varepsilon M \qquad 0 < \varepsilon << 1$$

$$x_M = -\varepsilon D \quad x_m = (1-\varepsilon)D$$



$$R \cong D\left(1 - \frac{8-\sqrt{2}}{12}\varepsilon\right)$$

$$2\theta \cong \pm 60^\circ$$

Lagrange Points (18):

$$m = \varepsilon M \quad 0 < \varepsilon << 1$$

$$x_M = -\varepsilon D \quad x_m = (1-\varepsilon)D$$

- Summary for zero *y*:
  - Between *M* and *m*: L1: $$x \cong D\left(1 - \sqrt[3]{\frac{1}{3}\varepsilon}\right)$$
  - On the opposite side of *m* from *M*: L2: $$x \cong D\left(1 + \sqrt[3]{\frac{1}{3}\varepsilon}\right)$$
  - On the opposite side of *M* from *m*: L3: $$x \cong -D\left(1 + \frac{5}{12}\varepsilon\right)$$

Lagrange Points (19):

$$m = \varepsilon M \quad 0 < \varepsilon << 1$$

$$x_M = -\varepsilon D \quad x_m = (1-\varepsilon)D$$

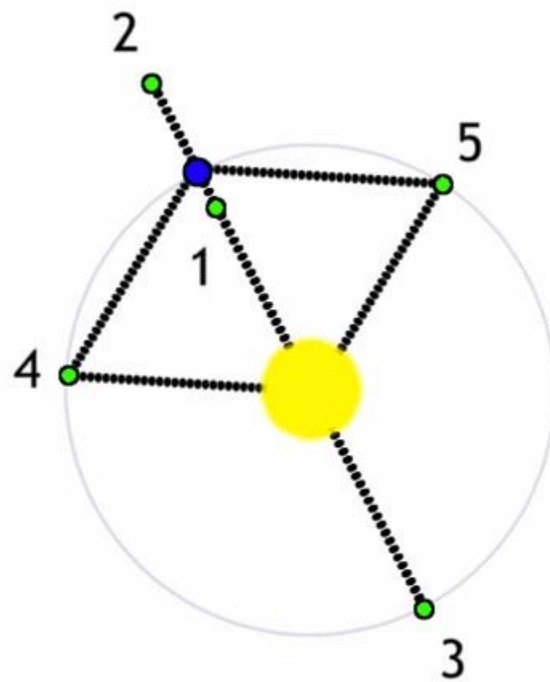
- Summary for nonzero *y*:
  - Ahead of *m* in orbit: L4:

$$2\theta \cong +60^\circ \qquad R \cong D\left(1 - \frac{8-\sqrt{2}}{12}\varepsilon\right)$$

  - Behind *m* in orbit: L5:

$$2\theta \cong -60^\circ \qquad R \cong D\left(1 - \frac{8-\sqrt{2}}{12}\varepsilon\right)$$

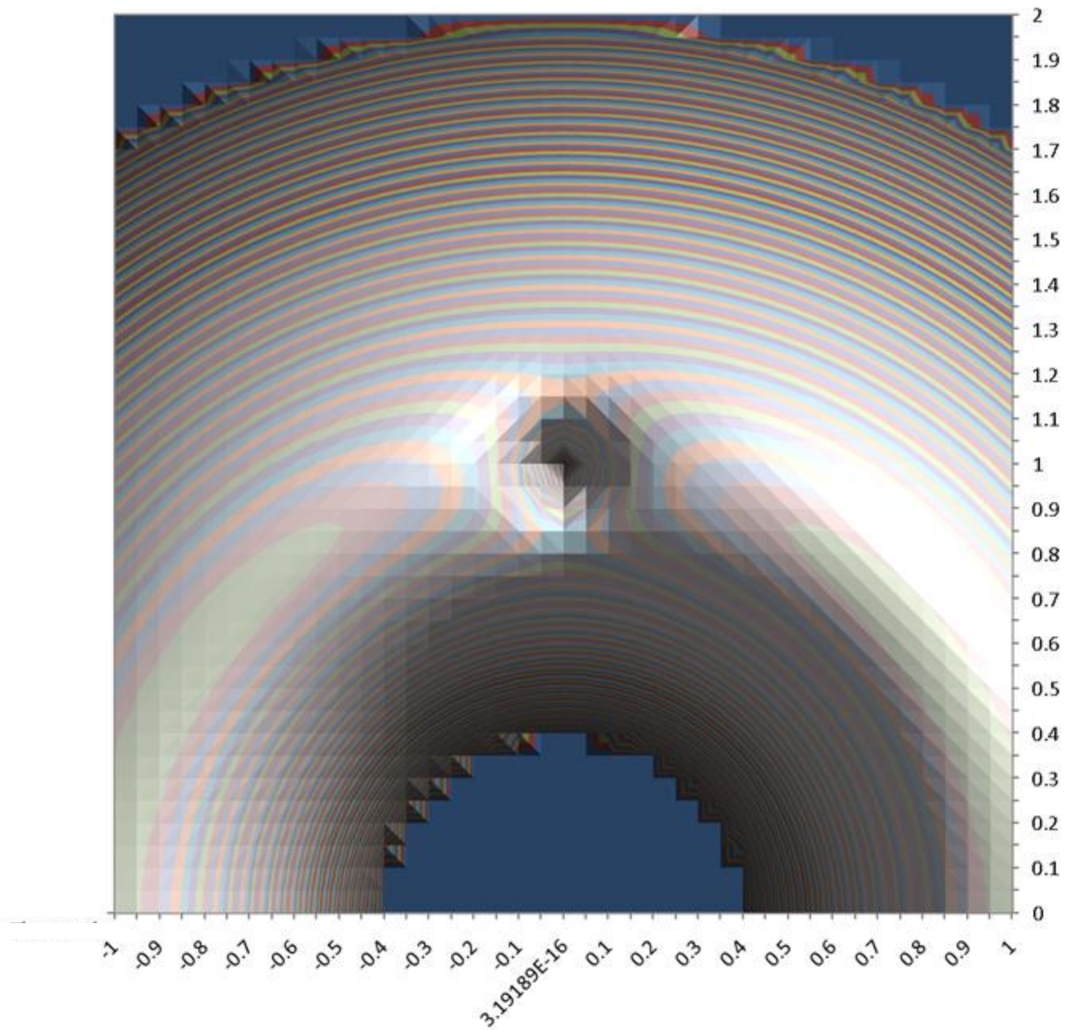
Summary of Lagrange Points



(What is wrong with this picture?)

(courtesy of Wikipedia, "Lagrange Points")

Potential Energy Contours



Stability of Lagrange Points

- Requires saving higher-order terms in linearized equations
- Results:
  - L1, L2, L3 unstable
    - Body perturbed from the point will move away from the point
  - L4, L5 neutrally stable
    - Body perturbed from the point will drift around the point

Conclusions

- Stability of Spinning Bodies
- Derivation of Lagrange Points